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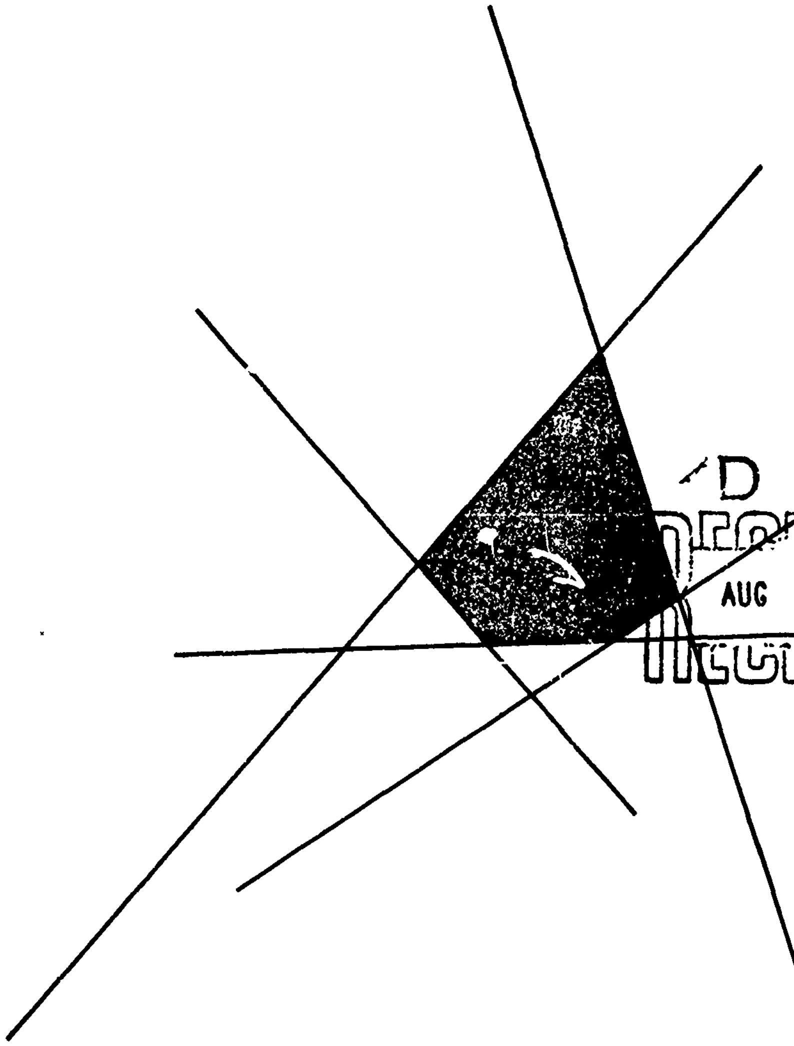
OPTIMAL SEARCH MODELS

by

YI CHI KAN



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Operations Research Center
University of California, Berkeley

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ABSTRACT

A target is located in one of n boxes. Initially, the target is in box i with a given prior probability p_i^o, $\sum p_i^o = 1$. A sequential search is made. Searching box i costs $c_i > 0$ and finds the target with probability α_i (i.e., the overlook probability is $1 - \alpha_i$) if the target is in the box at that time. A reward R_i is earned if the target is found in box i. A strategy is any rule for determining when to search, and if so, which box. The objective is to maximize the probability of finding the target in a given number of searches or to minimize the risk (expected searching cost minus expected reward).

In the above model, suppose $n = 2$ and the objective is to minimize the risk. Consider the optimal strategy as a function of the state (defined as the posterior probability vector). Let S_o be the set of states for which an optimal strategy stops searching. Let S_i be the set of states for which an optimal strategy searches box i, $i = 1,2$. A counterexample shows that although S_o is a convex set, surprisingly, S_i need not be convex.

A moving target model is studied in which a target is assumed to move from box to box in accordance with a Markov transition probability matrix. Conditions are given so that the optimal strategy can be characterized for a general n box model.

In an optimal search model with random overlook probabilities, the α_i's are allowed to be random variables. For instance, the α's may be random due to weather

condition. Let α_i^t be the α's at the t-th stage told after the t-th search. For fixed i, it is assumed that $\alpha_i^1, \alpha_i^2, \ldots$ are independent identically distributed random variables. The following results are derived. To maximize the probability of finding a target in a given number of searches, an optimal strategy searches at each time a box with $\max p_i E\alpha_i$. To minimize the expected searching cost before finding the target, an optimal strategy searches at each time a box with $\max \frac{p_i E\alpha_i}{c_i}$. Although these results resemble the classic results for a model with deterministic α_i's, the proofs are entirely new.

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CHAPTER 1

INTRODUCTION

1.1 Introduction of the Model

Optimal search models have been of theoretical interest as well as practical importance. In practical application, the most frequently encountered problem of this type would be the optimal search of a target. The target may be in any one of m regions, which is the same as saying a ball may be in any one of the m boxes as treated in this thesis. An optimal decision is desired as to which region to search in order to find or hit the target. Prior to the search, it is assumed that the probability distribution of the location of the target is known. Suppose further that, due to technical errors or other reasons, one might miss the target even when the correct location is searched. Thus after the search, if one misses the target, some information is gained and used for the next search. The problem is to find an optimal sequence of searches in order to maximize the probability of finding the ball in a finite number of searches or to minimize the expected searching cost before finding the target.

One can easily think of some possible complications of the above problem. For example, the target may be moving; the overlook probability may be random due to weather condition, etc. These are the various aspects of the problem which will be investigated in this thesis. A more precise mathematical model will be given later.

1.2 Background

The problem of optimal search models has been studied by many authors. Among them are Blackwell, Chew, Ross, Kadane, Pollock, etc.

A simple optimal search model is as follows. Suppose a ball is in one of m boxes. Initially, the ball is known to be in box i with probability P_i^o, $\sum_{i=1}^{m} P_i^o = 1$. A sequential search is made. Searching box i incurs a cost $c_i > 0$. The probability of finding the ball is α_i (i.e., the overlook probability is $1 - \alpha_i$) if a search is made in box i, given that the ball is in that box. After a search if the ball is found then the searching process terminates. If the ball is not found, then the searching process continues. The objective is to minimize the expected cost before finding the ball.

Blackwell (1962) characterized an optimal strategy for the above model. He showed that an optimal strategy is to search at any time that box with $\max \frac{\alpha_i p_i}{c_i}$, p_i being the posterior probability of the ball being in box i at that time.

Chew (1967) considered the case of equal costs and introduced the option of stopping at a penalty. He required at least one of the α's to be zero and proved that an optimal stopping rule exists. Some of the results he obtained are as follows:

1. An optimal strategy either stops or searches the box with $\max \alpha_i p_i$.
2. To maximize the probability of finding the ball in L searches, it is optimal to search the box with $\max \alpha_i p_i$.

Kadane (1968) considered the problem of maximizing the probability of finding the ball under a budget ceiling. He allowed the costs and overlook probabilities to depend on the number of searches made in a box. By applying the Neyman-Pearson Lemma, he proved that, under some conditions, it is optimal to search the box with maximum probability per cost.

Ross (1969) investigated a general optimal search and stop model. He assumed that a reward R_i is gained if a ball is found in box i. If $R_i \equiv R$, then this is equivalent to a penalty for stopping without finding the ball. He used a general result on negative dynamic programming to show that an optimal strategy exists. The main results he obtained are as follows.

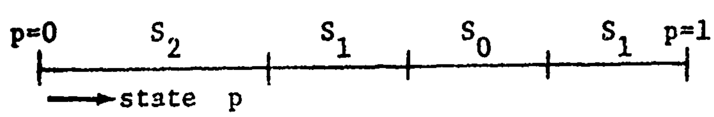
1. The optimal risk, defined as the expected searching cost minus the expected reward, is a concave function of the initial distribution $\mathbb{P} = \{p_i^o\}$. The optimal stopping region, defined as the set of P^o at which it is optimal to stop, is convex.
2. For the equal rewards but unequal costs case, he proved that an optimal strategy either stops or searches the box with $\max \frac{\alpha_i p_i}{c_i}$. That is more general than Chew's result, since costs are allowed to be different, and no requirements on α_i assumed.
3. For the case where both the rewards and the costs are allowed to be different, he proved that an optimal strategy either searches the box with $\max \frac{\alpha_i p_i}{c_i}$ or else never searches that box in the sequence that follows.

Pollock (1970) introduced the optimal search model of a moving target. He assumed that the target moves from box i to box j with probability p_{ij} after every search. Otherwise, the model is the same as previous ones. He took the model with two boxes and characterized the optimal strategy for the perfect detection case $(\alpha_i \equiv 1)$ and the no information case (i.e., the matrix p_{ij} has identical rows). For the general case, he incorrectly proved that an optimal initial decision, as

a function of the initial distribution, can be represented as two regions. This will be discussed in detail in Chapter 3.

1.3 Introduction of the Subsequent Chapters

The purpose of this thesis is to consider various versions of optimal search models. In Chapter 2, an optimal search and stop model with two boxes is considered. The rewards are assumed equal. Ross's results for this model as mentioned in the preceding paragraph have shown that the stopping region S_0 is convex. A natural question to ask is as follows. If S_i, $i = 1,2, \ldots$ is the set of states for which it is optimal to search box i, is it necessarily true that S_i be convex as well? Intuitively, one would say yes. However, the counterexample in Chapter 2 shows the contrary. It shows, in the case of two boxes, where the state variable can be represented as the probability that the ball is in box 1, the structure of an optimal policy is



where neither of the regions need be vacuous. Hence S_1 need not be convex.

In Chapter 3, Pollock's optimal search model of a moving target is undertaken. However, results such as for the no information case as well as the perfect detection case are generalized to n boxes' case. The proof is quite different. Next, some results on the model with a Jordan matrix as transition probability matrix are derived. When a stop option is added, the following result applies. If, in minimizing

the expected number of searches, it is optimal to search the box with $\max \alpha_i p_i$, then when one is allowed to stop, an optimal strategy either stops or searches the box with $\max \alpha_i p_i$. Finally, a two box optimal search model with $\alpha_1 = \alpha_2$ and symmetric transition probability matrix is studied in detail. Conditions are given to assure that searching the box with larger p_i is optimal. Also under some condition, the optimal strategy takes on an alternating searching sequence.

In Chapter 4, the overlook probabilities of an optimal search model are allowed to be random variables. Specifically, let α_i^t be the α's at the t^{th} stage. For fixed i , it is assumed that $\alpha_i^1, \alpha_i^2, \ldots$ are independent identically distributed random variables. Two cases may occur. First, at each stage, the random overlook probabilities are told after the search. For this case, the following results are derived. To maximize the probability of finding the ball in m searches, an optimal strategy searches the box with $\max_i p_i E\alpha_i^t$. To minimize the expected cost, an optimal strategy searches the box with $\max_i \frac{p_i E\alpha_i^t}{c_i}$. When α_i^t is deterministic and independent of t , this reduces to the classic results due to Blackwell and Chew. Secondly, at each stage, the random overlook probabilities are told before the search. For this case, it was hoped that under some restrictions, an optimal strategy would be similar to that in the first case. Unfortunately, this is not so, as demonstrated by a number of counterexamples.

CHAPTER 2

A COUNTEREXAMPLE FOR AN OPTIMAL SEARCH AND STOP MODEL

2.1 The Model

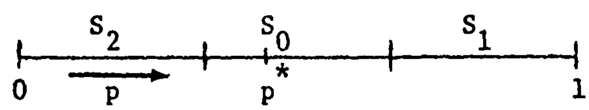
Consider the type of optimal search and stop model introduced by Ross [5]. Let there be two boxes. Let p_i^o be the given prior probability that a ball is hidden in box i, $i = 1,2, \sum p_i^o = 1$. A search of box i costs c_i $(c_i > 0)$ and finds the ball with probability α_i if the ball is in that box. Assume that a reward R is earned if the ball is discovered. At the beginning of each time period $t = 1,2, \ldots$ a searcher may decide to search box 1 or box 2 or to stop searching. The objective is to find an optimal strategy to maximize the expected net reward (expected reward minus expected searching cost).

Let the state at any time be characterized by p_i, $i = 1,2$ where p_i is the posterior probability that the ball is in box i at a certain time (or stage). Since there are only two boxes the state at any time can be represented by a scalar p, where $p_1 = p$, $p_2 = 1 - p$. Then the following results are due to Ross.

i) At any time t, an optimal strategy either searches a box with $\max_i \alpha_i p_i$ or else stops. In terms of the state p, this implies that there exists a number p^*, $0 \leq p^* \leq 1$, such that if $p \geq p^*$, an optimal strategy either stops or else searches box 1; if $p \leq p^*$, an optimal strategy either stops or else searches box 2. p^* is determined by $\dfrac{\alpha_1 p^*}{c_1} = \dfrac{\alpha_2(1 - p^*)}{c_2}$.

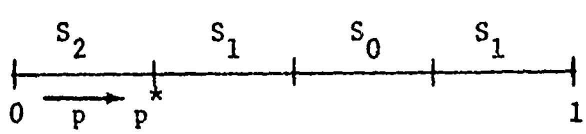
ii) The stopping region S_0, defined as the set of states for which it is initially optimal to stop is a convex region or an interval since p is a scalar.

Let the horizontal coordinate be p, $0 \leq p \leq 1$. Let S_i, $i = 1,2$ be the set of states for which it is initially optimal to search box i. Then the structure of the optimal policy is characterized by S_0, S_1, S_2 on p. In fact, by i) and ii), if $p^* \in S_0$, then there exists an optimal policy which has at most three regions as below.

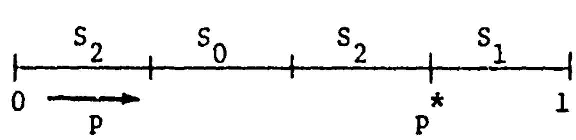


This policy of three regions is intuitive. It says that if p, the probability that the ball is in box 1 is large, then box 1 is searched; if p is small, meaning that $p_2 = 1 - p$ is large, then box 2 is searched. On the other hand, if p is somewhere in the middle, then stop.

At this point, one may raise the question: Could it happen that $p^* \notin S_0$? If so, then by i) and ii) the structure of the optimal strategy could be like



or



More precisely,

$$\left.\begin{array}{r} S_0 \ \text{nonempty} \\ p^* \notin S_0 \\ 1 \notin S_0 \\ 0 \notin S_0 \end{array}\right\} \Rightarrow \quad \text{an optimal policy which has four regions}$$

Contrary to intuition, the counterexample will show that the structure of four regions could occur. Basically, it says that one might want to search box 1 when p is large and stop when p is slightly smaller. Then when p is still smaller, surprisingly, one searches box 1 again before searching box 2.

2.2 The Counterexample

A strategy is any sequence (or partial sequence) $\delta = (\delta_1, \ldots, \delta_s)$ where $\delta_i \in \{1,2, \ldots, m\}$ for $i = 1, \ldots, s$ and $s \in \{0,1,2, \ldots, \infty\}$. The policy δ instructs the searcher to search box δ_i at the ith stage and to stop searching if the object has not been found after the sth search. $s = 0$ means that the searcher stops immediately and $s = \infty$ means that he does not stop until he finds the ball.

For any strategy δ and any state p, $0 \leq p \leq 1$, let

$f(p,\delta)$ = the expected net reward (expected reward minus expected searching cost) incurred when p is the prior probability that the ball is in box 1 and strategy δ is employed.

Let $f(p) = \sup\limits_{\delta} f(p,\delta)$. The following lemma will be used in the counterexample.

Lemma 2.1:

Let δ^0 , δ^1 , δ^2 be some searching strategies and δ^* be the strategy of searching $\max\limits_{1,2}\left\{\frac{\alpha_1 p}{c_1}, \frac{\alpha_2(1-p)}{c_2}\right\}$ until finding the ball. Then the following conditions imply that the structure of the optimal policy may have four regions.

$$f(1,\delta^1) > 0 \;,\; f(0,\delta^0) > 0 \;,\; f(p^*,\delta^2) > 0 \;,\; f(p^*,\delta^*) < 0 \;.$$

Proof:

$$f(1) \geq f(1,\delta^1) > 0 \Rightarrow 1 \notin S_o$$

$$f(0) \geq f(0,\delta^0) > 0 \Rightarrow 0 \notin S_0$$

$$f(p^*) \geq f(p^*,\delta^2) > 0 \Rightarrow p^* \notin S_0 \;.$$

Suppose the structure of the optimal policy has no stopping region, i.e., the optimal strategy never stops. Then clearly δ^* is optimal for all $p \in [0,1]$, which implies $f(p^*,\delta^*) \geq 0$. Therefore, $f(p^*,\delta^*) < 0$ implies that the stopping region S_0 is not vacuous. It follows from i) and ii) that there exists an optimal policy which has four regions

Q.E.D.

It remains to find numerical values for the parameters so that the conditions in the lemma are satisfied.

Let $R = 6.6$, $\alpha_1 = 3/4$ $\alpha_2 = 1/2$ $c_1 = 1$, $c_2 = 3$,

δ^0 = keep on searching box 2 until finding the ball.

δ^1 = keep on searching box 1 until finding the ball.

δ^2 = search box 1 then box 2 then stop.

δ^3 = the sequence used by following δ^*, given that the initial state is p^*.

Let $p^{(i)}$ be the posterior probability of the process after ith stage given that the initial state is p^* and that δ^* is used. At p^*, $\frac{\alpha_1 p^*}{c_1} = \frac{\alpha_2(1 - p^*)}{c_2}$. Hence δ^* says one may search either box 1 or box 2, i.e., $\delta_1 = 1$ or 2. Suppose $\delta_1 = 1$, then

$$p^* = \frac{\frac{\alpha_2}{c_2}}{\frac{\alpha_1}{c_1} + \frac{\alpha_2}{c_2}} = \frac{2}{11}$$

$$\frac{\alpha_1 p^{(1)}}{c_1} : \frac{\alpha_2(1 - p^{(1)})}{c_2} = (1 - \alpha_1)\,\frac{\alpha_1 p^*}{c_1} : \frac{\alpha_2(1 - p^*)}{c_2}$$

$$= 1 - \alpha_1 : 1 \Rightarrow \delta_2 = 2$$

$$\frac{\alpha_1 p^{(2)}}{c_1} : \frac{\alpha_2(1 - p^{(2)})}{c_2} = (1 - \alpha_1) : (1 - \alpha_2) = 1/4 : 1/2$$

$$\Rightarrow \delta_3 = 2$$

$$\frac{\alpha_1 p^{(3)}}{c_1} : \frac{\alpha_2[1 - p^{(3)}]}{c_2} = (1 - \alpha_1) : (1 - \alpha_2)^2 = 1$$

$$\Rightarrow \delta_4 = 1 \text{ or } 2 \Rightarrow p^{(3)} = p^* .$$

It follows that δ^3 can be a periodic sequence, namely $\delta^3 = 122,122, \ldots$. Consequently

$$\begin{aligned} f(p^*,\delta^*) &= f(p^*,\delta^3) \\ &= \alpha_1 p^* R - c_1 + \left(1 - \alpha_1 p^*\right)\left[\alpha_2(1 - p^{(1)})R - c_2\right] \\ &+ \left(1 - \alpha_1 p^*\right)\left[1 - \alpha_2(1 - p^{(1)})\right]\left[\alpha_2(1 - p^{(2)})R - c_2\right] \\ &+ \left(1 - \alpha_1 p^*\right)\left[1 - \alpha_2(1 - p^{(1)})\right]\left[1 - \alpha_2(1 - p^{(2)})\right]\cdot f(p^*,\delta^3) . \end{aligned}$$

Thanks to the recursive relation, one can compute $f(p^*,\delta^*)$ by substituting the numerical values of the parameters.

$$f(p^*,\delta^*) = \frac{33R - 218}{44} = 3/4\ (6.6 - 6.606 \ldots) < 0$$

$$f(1,\delta^1) = R - \frac{c_1}{\alpha_1} = 6.6 - 4/3 > 0$$

$$f(0,\delta^0) = R - \frac{c_2}{\alpha_2} = 6.6 - 6 > 0$$

$$\begin{aligned} f(p^*,\delta^2) &= \alpha_1 p^* R - c_1 + \left(1 - \alpha_1 p^*\right)\left[\alpha_2(1 - p^{(1)})R - c_2\right] \\ &+ \left(1 - \alpha_1 p^*\right)\left[1 - \alpha_2(1 - p^{(1)})\right]\left[\alpha_2(1 - p^{(2)})R - c_2\right] \\ &= \frac{24}{44}(6.6 - 6.583 \ldots) > 0 . \end{aligned}$$

Thus all the conditions in the lemma are satisfied, and the counterexample is complete.

CHAPTER 3

OPTIMAL SEARCH OF A MOVING TARGET

3.1 Introduction and Formulation

Let there be n boxes. A target is initially in box i with a given probability p_i, where $p_i \geq 0, \sum p_i = 1$. Then at discrete time (or stage) $t = 1,2, \ldots,$ it moves from box to box. If at time t, the target is in box i, then it will be in box j with probability p_{ij} at time $t + 1$ where $T = [p_{ij}]$, $1 \leq i \leq n$, $1 \leq j \leq n$, is a Markov transition probability matrix.

A sequential search is made. At each time t, a decision is made as to which box to search. The searching process continues until the target is found or until one decides to stop when there is a stop option. Searching box i incurs a cost $c_i > 0$ and finds the target with probability α_i if the target is in box i at that time, (i.e., $\beta_i = 1 - \alpha_i$ is the overlook probability for the ith box). Let R be the reward earned when the target is found. This is needed only when there is a stop option.

The objective is to maximize the probability of finding the target in a given number of searches or to minimize the expected net searching cost (expected searching cost minus expected reward).

Let state $\mathbb{P}$ be defined as the vector of posterior probabilities. $\mathbb{P} = (p_1, \ldots, p_n)$ where p_i is the probability that the target is in box i at that time.

A strategy is any rule for determining when to search, and, if so, which box. It is a sequence $\delta = (\delta_1, \ldots, \delta_s)$ where $\delta_i \in \{1,2, \ldots, n\}$ for $i = 1, \ldots, s$ and $s \in \{0,1,2, \ldots, \infty\}$. The policy δ instructs the searcher to search box δ_i at the ith stage

and to stop searching if the object has not been found after the sth search. $s = 0$ means that the searcher stops immediately; $s = \infty$ means that he does not stop until he finds the target. When there is no stop option, $s = \infty$.

For any strategy δ , any state $\mathbb{P}$ and any integer m , define the following functions:

$f^m(\mathbb{P},\delta)$ = the probability of finding the target in m searches when $\mathbb{P}$ is the initial state and strategy δ is employed.

$f_i^m(\delta)$ = the conditional probability of finding the target given that the target is initially in box i and strategy δ is employed.

$g(\mathbb{P},\delta)$ = the expected net searching cost when $\mathbb{P}$ is the initial state and strategy δ is employed.

$$f^m(\mathbb{P}) = \sup_\delta f^m(\mathbb{P},\delta)$$

$$g(\mathbb{P}) = \inf_\delta g(\mathbb{P},\delta)$$

Note that $f^m(\mathbb{P},\delta) = \sum_i p_i f_i^m(\delta)$.

Let $T_i\mathbb{P} = [(T_i,\mathbb{P})_1, \ldots, (T_i\mathbb{P})_n]$, $i = 1,2, \ldots, n$, where $(T_i\mathbb{P})_j$ is the posterior probability that the target is in box j at the next stage given that a present search of box i has not uncovered it.

Let $\mathbb{P}^i = \left[\mathbb{P}_1^i, \mathbb{P}_2^i, \ldots, \mathbb{P}_n^i\right]$, $i = 1,2, \ldots, n$, where $\mathbb{P}_j^i$ is the posterior probability that the target was in box j prior to the search given that a present search of box i has not uncovered it. Then

$T_i \mathbb{P} = \mathbb{P}^i \cdot T$ where T is the transition probability matrix

$$\mathbb{P}^i_j = \begin{cases} p_j(1 - \alpha_i p_i)^{-1} & (j \neq i) \\ (1 - \alpha_i)p_i(1 - \alpha_i p_i)^{-1} & (j = i) \, . \end{cases}$$

Theorem 3.1:

In the moving target model, let state $\mathbb{P} = (p_1, \ldots, p_n)$ be given. Suppose there is a box i such that

$$\alpha_i p_i p_{ik} \geq \alpha_j p_j p_{jk} \qquad \forall \; k, j \, .$$

Then to maximize the probability of finding the target in m searches, where m is any given number, an optimal strategy first searches box i.

Proof:

Let δ be any strategy. For any box j, let $S_j\delta$ be the strategy that searches first box j then follows the strategy δ.

Let box i be the same as defined in the theorem. If one can show

$$f^m(\mathbb{P}, S_i\delta) \geq f^m(\mathbb{P}, S_j\delta)$$

for any j, any strategy δ, then the theorem is proven. Now

$$\begin{aligned} f^m(\mathbb{P}, S_j\delta) &= \alpha_j p_j + (1 - \alpha_j p_j) f^{m-1}(\mathbb{P}^j T, \delta) \\ &= \alpha_j p_j + (1 - \alpha_j p_j) \sum_k (\mathbb{P}^j T)_k f_k^{m-1}(\delta) \end{aligned}$$

$$= \alpha_j p_j + \sum_k \left[\sum_{r\neq j} p_r p_{rk} + (1 - \alpha_j) p_j p_{jk}\right] f_k^{m-1}(\delta)$$

$$= \alpha_j p_j + \sum_k \left[\sum_r p_r p_{rk} - \alpha_j p_j p_{jk}\right] f_k^{m-1}(\delta)$$

$$= \sum_k \alpha_j p_j p_{jk}\left[1 - f_k^{m-1}(\delta)\right] + \sum_k \sum_r p_r p_{rk} f_k^{m-1}(\delta)$$

$$\leq \sum_k \alpha_i p_i p_{ik}\left[1 - f_k^{m-1}(\delta)\right] + \sum_k \sum_r p_r p_{rk} f_k^{m-1}(\delta)$$

$$= f^m(\mathbb{P}, S_i \delta)$$

Q.E.D.

To illustrate the use of the above theorem, the following facts are noticed.

1. If $p_i = 1$, $p_j = 0$ $j \neq i$, then an optimal strategy first searches box i.
2. If $p_{jk} = v_k$ $\forall$ j, then an optimal strategy first searches a box with $\max_i \alpha_i p_i$. This is the no information case and will be treated later in more detail.

3.2 Some Special Cases of the Moving Target Model

In this section, two special cases of the model will be exploited. Consider first the case where $p_{ij} = v_j$ $\forall$ i. This is the case where no information is gained from the previous stage. Let $V = (v_1, v_2, \ldots, v_n)$.

Theorem 3.2:

a) To maximize the probability of finding the target in m searches, where m is any given positive integer, an optimal strategy searches at each stage the box with $\max_i \alpha_i p_i$ where p_i is the posterior probability that the target is in box i at that stage. The maximized $f^m(\mathbb{P})$ is

$$f^m(\mathbb{P}) = 1 - \left(1 - \max_i \alpha_i p_i\right)(1 - \max_i \alpha_i v_i)^{m-1} .$$

b) To minimize the expected searching cost before finding the target, an optimal strategy, as well as the minimized expected searching cost can be determined by

$$g(\mathbb{P}) = \min_i [c_i + (1 - \alpha_i p_i) g(V)]$$

$$g(V) = \min_j \frac{c_j}{\alpha_j v_j}$$

(assuming not all $\alpha_j v_j$ are zero).

Proof:

a) Since $p_{ij} = v_j \ \forall \ i$, the posterior probability vector for the next stage given that a present search has not uncovered is $V = (v_1, \ldots, v_n)$. Hence

$$\begin{aligned} f^m(\mathbb{P}) &= \max_i \left[\alpha_i p_i + (1 - \alpha_i p_i) f^{m-1}(V)\right] \\ &= \max_i \left[\alpha_i p_i (1 - f^{m-1}(V))\right] + f^{m-1}(V) . \end{aligned}$$

Since $1 - f^{m-1}(V) \geq 0$, searching the box with $\max_i \alpha_i p_i$ is optimal and part (a) is proven.

b)

$$g(\mathbb{P}) = \min_i [c_i + (1 - \alpha_i p_i) g(V)]$$

$$g(V) = \min_i [c_i + (1 - \alpha_i v_i) g(V)] .$$

To minimize $g(V)$, one can simply solve the second equation for different i's . Thus

$$g(V) = \min_i \frac{c_i}{\alpha_i v_i}$$

Q.E.D.

Consider now the case where $\alpha_i \equiv 1$. This is called the perfect detection case. The following theorem applies.

Theorem 3.3:

In the moving target model, assume $\alpha_i \equiv 1$. To maximize the probability of finding the target in m searches, suppose an optimal strategy first searches box i at state $\mathbb{P} = (p_1, \ldots, p_n)$. Then the same is true at state $\mathbb{P}' = (p_1', \ldots, p_n')$ if $p_i' > p_i$, and p_j' is proportionally decreased $\forall\ j \neq i$. That is,

$$p_i' \geq p_i$$

$$p_j' = \lambda p_j \quad \text{where} \quad 0 \leq \lambda = \frac{1 - p_i'}{1 - p_i} \leq 1 \ .$$

Proof:

For any strategy δ , any box j , let $S_j\delta$ be the strategy that first searches box j and then follows strategy δ . Then, for any state $\mathbb{P}$, conditioning on the initial location of the target yields

$$f^m(\mathbb{P}, S_j\delta) = p_j + \sum_{k \neq j} p_k f_k^{m-1}(\delta) \ .$$

Notice that if the target is not in box j initially, then the probability of finding it depends on δ only.

Let box i be the same as given in the theorem. By assumption,

the strategy that searches box i first and then follows an optimal strategy, say δ^*, will be optimal for state $\mathbb{P}$, i.e.,

$$f^m(\mathbb{P}) = f^m\left(\mathbb{P}, S_i\delta^*\right).$$

Let box k be any box $k \neq i$. At state $\mathbb{P}'$ as given in the theorem, let $S_k\delta'$ be the strategy that searches box k first and then follows an optimal strategy δ'. If one can prove

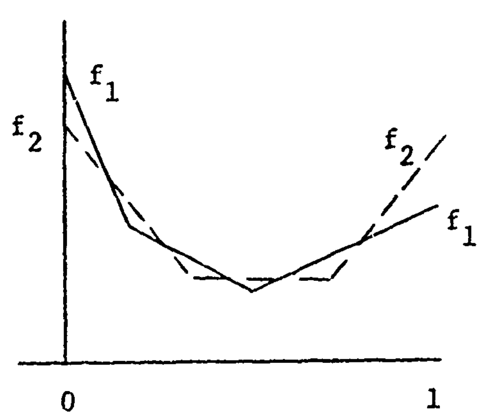
$$f^m\left(\mathbb{P}, S_i\delta^*\right) - f^m(\mathbb{P}', S_k\delta') \geq 0 ,$$

then there exists an optimal strategy which first searches box i at state $\mathbb{P}'$. Now

$$\begin{aligned}
& f^m\left(\mathbb{P}', S_i\delta^*\right) - f^m(\mathbb{P}', S_k\delta') \\
\geq\ & f^m\left(\mathbb{P}', S_i\delta^*\right) - f^m(\mathbb{P}', S_k\delta') - \lambda\left[f^m\left(\mathbb{P}, S_i\delta^*\right) - f^m(\mathbb{P}, S_k\delta')\right] \\
=\ & f^m\left(\mathbb{P}', S_i\delta^*\right) - \lambda f^m\left(\mathbb{P}, S_i\delta^*\right) - \left[f^m(\mathbb{P}', S_k\delta') - \lambda f^m(\mathbb{P}, S_k\delta')\right] \\
=\ & p_i' + \sum_{j\neq i} p_j' f_j^{m-1}(\delta^*) - \lambda p_i - \lambda \sum_{j\neq i} p_j f_j^{m-1}(\delta^*) \\
& - \left[p_k' + \sum_{j\neq k} p_j' f_j^{m-1}(\delta') - \lambda p_k - \lambda \sum_{j\neq k} p_j f_j^{m-1}(\delta')\right] \\
=\ & p_i' - \lambda p_i - \left[p_i' f_i^{m-1}(\delta') - \lambda p_i f_i^{m-1}(\delta')\right] \\
=\ & (p_i' - \lambda p_i)\left[1 - f_i^{m-1}(\delta')\right] \\
\geq\ & 0
\end{aligned}$$

Q.E.D.

Pollock (1970) analyzed the model with two boxes for the above two cases. He characterized the optimal strategy as a function of the initial probability distribution as follows. If p and $1 - p$ are respectively the probability that the target is in box 1 and box 2, then an optimal strategy is to search box 1 when $p \geq p^*$ and search box 2 when $p \leq p^*$. p^* can be explicitly computed. For the general case of the two box model (i.e., with no restriction on either α_i or T), he claimed a similar result holds but that p^* remains to be determined. He gave an incorrect proof. The proof was based on the implicit assertion that two convex functions f_1, f_2 on the real interval $[0,1]$ intersect at only one point if $f_1(0) > f_2(0)$, $f_1(1) < f_2(1)$. This is clearly wrong. Consider the following two functions f_1 and f_2 on the real line. They satisfy the above conditions but they may intersect at any odd number of points.



Comment: In the general case of the two box model, it is unknown yet whether an optimal strategy as a function of p has only two regions.

In the rest of this chapter, some more special cases will be studied.

3.3 The Moving Target Model with Some Transition Probability Matrices Related to a Jordan Matrix

A matrix is a Jordan matrix if there exists exactly one 1 in each row and each column, whereas all the rest of the elements are zero.

In the moving target model, if T is a Jordan matrix, it means the following. At time t, if the target is in box i, then at time $t+1$ it moves to box $h(i)$ with probability one, $i = 1, \ldots, n$, $h(i) = 1, 2, \ldots, n$. But $h(i) \neq h(j)$ for $i \neq j$. If, in addition, $\alpha_i \equiv \alpha \ \forall \ i$, then after every search, it appears as though the target is stationary but that the boxes are renumbered. Note that if $T = I$, the identity matrix, then the target is stationary.

For any strategy δ, any state $\mathbb{P}$, any transition probability matrix T, and any integer m, let

$g^m(\mathbb{P}, T; \delta)$ = the probability of finding the target in m searches when T is the transition probability matrix for the model, $\mathbb{P}$ is the initial state and strategy δ is employed.

Let

$$g^m(\mathbb{P}, T) = \sup_{\delta} g^m(\mathbb{P}, T; \delta) .$$

Let

$g_i^m(T;\delta)$ = the conditional probability of finding the target in m searches, given that the target is in box i initially, the transition probability matrix is T and strategy δ is employed.

Then

$$g^m(\mathbb{P},T;\delta) = \sum p_i g_i^m(T;\delta) \ .$$

Theorem 3.4:

Suppose T is a Jordan matrix, $\alpha_i \equiv \alpha$. Then, to maximize the probability of finding the target in N searches, where N is any given number, an optimal strategy searches the box with $\max_i p_i$ each time where p_i is the posterior probability that the target is in box i at that time. Also

$$g^N(\mathbb{P},T) = g^N(\mathbb{P},I)$$

where I is identity matrix, i.e., when

$$\alpha_i \equiv \alpha \ ,$$

the maximized probability of finding the target in N searches when T is a Jordan matrix is the same as when the target is stationary.

To prove the theorem, induction will be used. $N = 1$ is trivial. It will be verified that if the theorem holds for $N = m - 1$ then it holds for $N = m$ as well. Now

$$g^m(\mathbb{P},T) = \max_i \alpha p_i + (1 - \alpha p_i) g^{m-1}(T_i\mathbb{P},T) \ .$$

By induction hypothesis

$$g^{m-1}(T_i\mathbb{P},T) = g^{m-1}(T_i\mathbb{P},I) \ .$$

By definition, $T_i\mathbb{P} = \mathbb{P}^iT$. Hence

$$g^{m-1}(T_i\mathbb{P},I) = g^{m-1}(\mathbb{P}^iT,I) \ .$$

$g^{m-1}(\mathbb{P}^iT,I)$ corresponds to the probability function for a $m - 1$ stage search model where the target is stationary, $\alpha_i \equiv \alpha$, and the initial state is $\mathbb{P}^iT$. $\mathbb{P}^i$ multiplied by T means nothing but a renumbering of the boxes where all the boxes are identical $(\alpha_i \equiv \alpha)$. Therefore,

$$g^{m-1}(\mathbb{P}^iT,I) = g^{m-1}(\mathbb{P}^i,I)$$

and

$$g^m(\mathbb{P},T) = \max_i \left[\alpha p_i + (1 - \alpha p_i) g^{m-1}(\mathbb{P}^i,I)\right] \ .$$

A look at the definition of $\mathbb{P}^i$ shows that the right-hand side is just the formulation for a m stage stationary target model. Hence searching a box with $\max_i p_i$ is optimal and

$$g^m(\mathbb{P},T) = g^m(\mathbb{P},I)$$

Q.E.D.

Theorem 3.5:

For any integer m, any prior probability vector $\mathbb{P}$, and any transition probability T, $g^m(\mathbb{P},T)$ is a convex function of $\mathbb{P}$.

Proof:

Let $\mathbb{P}^1 = \left(p_1^1, \ldots, p_n^1\right)$ $\mathbb{P}^2 = \left(p_1^2, \ldots, p_n^2\right)$ be any two prior probability vectors. Let λ be any number $0 \leq \lambda \leq 1$. Then

$$
\begin{aligned}
& g^m[\lambda\mathbb{P}^1 + (1-\lambda)\mathbb{P}^2, T] \\
&= \sup_\delta g^m[\lambda\mathbb{P}^1 + (1-\lambda)\mathbb{P}^2, T;\delta] \\
&= \sup_\delta \sum [\lambda\mathbb{P}^1 + (1-\lambda)\mathbb{P}^2]_i g_i^m(T,\delta) \\
&\leq \lambda \sup_\delta \sum_i p_i^1 g_i^m(T,\delta) + (1-\lambda) \sup_\delta \sum p_i^2 g_i^m(T,\delta) \\
&= \lambda\, g^m(\mathbb{P}^1,T) + (1-\lambda) g^m(\mathbb{P}^2,T) .
\end{aligned}
$$

Hence, $g^m(\mathbb{P}, T)$ is a convex function of $\mathbb{P}$.

Q.E.D.

The following theorem gives an upper bound for the maximum probability of finding the target in N searches for a large class of transition probability matrices when $\alpha_i \equiv \alpha$.

Theorem 3.6:

Let T be a convex combination of Jordan matrices. Assume $\alpha_i \equiv \alpha$. Then $g^N(\mathbb{P},T) \leq g^N(\mathbb{P},I)$ where I is the identity matrix.

<u>Proof:</u>

Induction will be used. When $N = 1$, $g^1(\mathbb{P},T) = g^1(\mathbb{P},I) = \max_i \alpha p_i$. The theorem clearly holds.

It will be verified that if the theorem holds for $N = m - 1$, then it holds for $N = m$ as well. Now

$$g^m(\mathbb{P},T) = \max_i \alpha p_i + (1 - \alpha p_i) g^{m-1}(T_i\mathbb{P},T) .$$

By assumption, T is a convex combination of Jordan matrices. Hence, T can be written as $T = \sum a_i Q^i$ where Q^i , $i = 1,2,\ldots$ are Jordan matrices and a_i , $i = 1,2,\ldots$ are such that $a_i \geq 0, \sum a_i = 1$.

By induction hypothesis,

$$g^{m-1}(T_i\mathbb{P},T) \leq g^{m-1}(T_i\mathbb{P},I) .$$

By definition,

$$g^{m-1}(T_i\mathbb{P},I) = g^{m-1}(\mathbb{P}^iT,I) .$$

$g^{m-1}(T_i\mathbb{P},I)$ is a convex function of $T_i\mathbb{P}$, by the preceding theorem. Hence

$$\begin{aligned} g^{m-1}(\mathbb{P}^iT,I) &= g^{m-1}\left(\mathbb{P}^i \sum_k a_k Q^k, I\right) \\ &\leq \sum_k a_k g^{m-1}(\mathbb{P}^i Q^k, I) . \end{aligned}$$

Since Q^k is a Jordan matrix

$$g^{m-1}(\mathbb{P}^{i}Q^{k},I) = g^{m-1}(\mathbb{P}^{i},I)$$

by the same arguments as used in the proof of Theorem 3.4. Therefore,

$$g^{m-1}(T_i\mathbb{P},I) \leq g^{m-1}(\mathbb{P}^{i},I)$$

and

$$g^{m}(\mathbb{P},T) \leq \max\left[\alpha p_i + (1 - \alpha p_i)g^{m-1}(\mathbb{P}^{i},I)\right] .$$

But the right-hand side is just the maximized probability of a m stage stationary target model. Hence

$$g^{m}(\mathbb{P},T) \leq g^{m}(\mathbb{P},I)$$

Q.E.D.

3.4 Moving Target Model with a Stop Decision

Consider a problem where one may decide to stop before finding the target. Assume $c_i = c$ for all i. Let R be the reward earned when the target is found. The objective is to maximize the expected net return (i.e., expected reward minus expected searching cost). Call this problem (C). The following theorem applies.

Theorem 3.7:

Let $\mathbb{P} = (p_1, \ldots, p_n)$ be posterior probability vector. Suppose in the problem of maximizing the probability of finding the target in a given number of searches, an optimal strategy searches at each time a box with $\max\limits_i \alpha_i p_i$. Then in problem (C) an optimal strategy either first searches a box with $\max\limits_i \alpha_i p_i$ or else stops. Note that this applies

to the no information case in Theorem 3.2, the case in Theorem 3.4 and the stationary target case.

<u>Proof</u>:

Let δ be an optimal strategy. Its existence can be proven as in Ross's paper [6]. Let s be the time at which the searcher stops if the target has not been found after the sth search. s is deterministic and may be infinite. Let N_o be the time at which the target is found. $N_o = \infty$ if the target is not found. N_o is a random variable.

For any positive integer $i \leq s$, let $P_\delta(N_o < i)$, $P_\delta(N_o = i)$ be respectively the probabilities that the target is found before and at the ith search when strategy δ is employed. $P_\delta(N_o \geq i)$ is the probability that the target is not found in the first $i - 1$ searches by using strategy δ.

If $s = \infty$, then since δ is optimal, the target is found with probability 1. Otherwise, the expected searching cost will be infinity and δ cannot be optimal. Thus, if $s = \infty$, the expected net reward is

$$R \cdot P_\delta(N_o < \infty) - c \cdot \sum_{k=1}^{\infty} k P_\delta(N_o = k)$$

$$= R - c \cdot \sum_{k=1}^{\infty} P_\delta(N_o \geq k) .$$

If $0 < s < \infty$, the expected net reward is

$$R \cdot P\,(N_o \le s) - c \cdot \left[\sum_{k=1}^{s-1} kP_\delta(N_o = k) + S \cdot P\,(N_o \ge s) \right]$$

$$= R[1 - P_\delta(N_o \ge s + 1)] - c \cdot \sum_{k=1}^{s} P_\delta(N_o \ge k) \; .$$

In either case, if $s > 0$ (i.e., not stopping immediately), the strategy of searching a box with $\max_i \alpha_i p_i$ will, by assumption, minimize $P_\delta[N_o \ge k] \;\; \forall \; k$. Hence, an optimal strategy for problem (C) either first searches a box with $\max_i \alpha_i p_i$ or else stops.

3.5 A Special Case with Two Boxes

Consider a rather special case of the moving target model. Assume there are two boxes and $\alpha_1 = \alpha_2 = \alpha$. The objective is to maximize the probability of finding the target in N searches, where N is any given number. Given initial state $\mathbb{P} = (p_1, p_2)$. T^1 is the symmetric transition probability matrix such that

$$T^1 = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \text{ where } a \, , \, b \ge 0 \, , \, a + b = 1 \; .$$

An optimal strategy for the above problem will be characterized in the following theorems.

Theorem 3.8:

An optimal strategy for the above problem first searches the same box as a similar problem with $\alpha_1 = \alpha_2 = \alpha$ but with a transposed transition probability matrix

$$T^2 = \begin{bmatrix} b & a \\ a & b \end{bmatrix} \text{ where } a \text{ and } b \text{ are the same as in } T^1 \; .$$

Also the maximum probabilities of finding the target are equal for the two problems, i.e., $g^N(\mathbb{P},T^1) = g^N(\mathbb{P},T^2)$.

Remark:

This theorem implies that one can assume $a \geq b$ in T^1 in deciding which box to search first.

Proof:

Induction will be used. $N = 1$ is trivial. It will be verified that if the theorem holds for $N = m - 1$ then it holds for $N = m$ as well. Now for $k = 1,2$

$$g^m(\mathbb{P},T^k) = \max_i \left[\alpha p_i + (1 - p_i) g^{m-1}(\mathbb{P}^i T^k, T^k)\right]$$

where $\mathbb{P}^i = \left(\mathbb{P}^i_1, \mathbb{P}^i_2\right)$ as before. Recall that

$$\mathbb{P}^i_j = \begin{cases} p_j(1 - \alpha_i p_i)^{-1} & (j \neq i) \\ (1 - \alpha_i)p_i(1 - \alpha_i p_i)^{-1} & (j = i) \ . \end{cases}$$

By induction hypothesis

$$g^{m-1}(\mathbb{P}^i T^1, T^1) = g^{m-1}(\mathbb{P}^i T^1, T^2) \ .$$

By definition $T^2 = T^1 Q$ where $Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Multiplying the posterior probability by Q means a renumbering of the boxes. The two boxes have the same parameters, namely, the same overlook probability as well as the same probability of moving to the other box after one search. Therefore, by symmetry

$$g^{m-1}(\mathbb{P}^{\,i}T^1Q,T^1) = g^{m-1}(\mathbb{P}^{\,i}T^1,T^1) \ .$$

It follows that

$$g^{m-1}(\mathbb{P}^{\,i}T^2,T^2) = g^{m-1}(\mathbb{P}^{\,i}T^2,T^1) = g^{m-1}(\mathbb{P}^{\,i}T^1,T^1) \ .$$

Hence, by the above dynamic programming formulation, an optimal strategy for both problems first searches the same box and $g^m(\mathbb{P},T^1) = g^m(\mathbb{P},T^2)$.

Q.E.D.

<u>Theorem 3.9</u>:

If $\mathbb{P}$ is such that $\frac{p_1}{p_2} \geq \frac{a}{b}$ or $\frac{p_1}{p_2} \leq \frac{b}{a}$ where $a \geq b \geq 0$ as before then an optimal strategy first searches the box with larger p_i .

<u>Proof</u>:

Since $p_{11} = p_{22} = a$, $p_{12} = p_{21} = b$

$$\frac{p_1}{p_2} \geq \frac{a}{b} \Rightarrow \alpha p_1 p_{12} = \alpha p_1 b \geq \alpha p_2 a = \alpha p_2 p_{22} \ .$$

Also

$$\frac{p_1}{p_2} \geq \frac{a}{b} \geq 1 \Rightarrow \alpha p_1 p_{11} = \alpha p_1 a \geq \alpha p_2 b = \alpha p_2 p_{21} \ .$$

Hence by Theorem 3.1, an optimal strategy first searches box 1 (i.e., the box with larger p_i). By symmetry, the case $\frac{p_1}{p_2} \leq \frac{b}{a}$ is the same.

Q.E.D.

Theorem 3.10:

Assume $a \geq b$ in T^1 and state $\mathbb{P}$ is such that $1 - \alpha \leq p_1/p_2 \leq 1 - \alpha$. Then, to maximize the probability of finding the target in N searches, where N is any given integer, an optimal strategy is as follows. It first searches a box with larger p_i, and then keep on switching to the other box after every search.

Proof:

The transition probability matrix is

$$T = T^1 = \begin{bmatrix} a & b \\ b & a \end{bmatrix} = (a - b) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2b \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= q_1 A_1 + q_2 A_2 \quad \text{where} \quad q_1 = a - b \geq 0 \text{ , } q_2 = 2b \geq 0 \text{ ,}$$

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \text{ , } A_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} .$$

One can consider the target moves after every search in the following way. With probability q_1 it moves according to A_1 (i.e., it stays). With probability $q_2 = 1 - q_1$, it moves according to A_2. Physically, this doesn't occur. But one may think of the ball moves in the above fashion even after the ball is found.

Let r be the first time it moves according to A_2, $1 \leq r \leq N$. $r = 1$ means that after one search it moves according to A_2 for the first time. $r = N$ means it always moves according to A_1. r is independent of the strategy.

Clearly, r is a random variable with the following distribution. Let t be an integer $1 \leq t \leq N$. Then

$$r = t \text{ with probability } q_1^t q_2 \text{ if } 1 \le t \le N-1$$

$$r = t \text{ with probability } q_1^N \text{ if } t = N .$$

Let strategies δ^1 , δ^2 be defined as follows.

$$\delta^1 = 1\ 2\ 1\ 2\ \ldots$$

$$\delta^2 = 2\ 1\ 2\ 1\ \ldots .$$

That is, δ^1 and δ^2 are alternating searching sequences before the process ends.

Assume first $p_1 \ge p_2$. Then the theorem says that δ^1 is optimal if $1 - \alpha \le \frac{p_1}{p_2} \le \frac{1}{1-\alpha}$. The case $p_2 \ge p_1$ is similar and omitted. To prove the theorem, induction will be used. $N = 1$ is trivial. It will be verified that if the theorem holds for $N \le m - 1$, then it holds for $N = m$ as well.

Recall that for any positive integer m , $g^m(\mathbb{P},T;\delta)$ is the probability of finding the target in m searches when $\mathbb{P}$ is the state, T is the transition probability vector and strategy δ is employed.

Let V be the state $(\frac{1}{2},\frac{1}{2})$. Let δ^0 be the truncated δ^1 after truncating the partial sequence for the first r stages. Conditioning on r yields:

$$\begin{aligned} g^m(\mathbb{P},T;\delta^1) &= E\left\{g^r\left(\mathbb{P},A_1;\delta^1\right) + \left[1 - g^r\left(\mathbb{P},A_1;\delta^1\right)\right]g^{m-r}(V,T;\delta^0)\right\} \\ &= 1 - E\left[1 - g^r\left(\mathbb{P},A_1;\delta^1\right)\right][1 - g^{m-r}(V,T;\delta^0)] . \end{aligned}$$

The above formulation can be explained as follows. During the first r stages, the target moves according to A_1 . Therefore, the probability

of finding the target in the first r searches is $g^r\left(\mathbb{P}, A_1, \delta^1\right)$. At the end of the rth search, if the target is not found, then it moves according to A_2. Hence the state for the $r + 1$st stage is $V = (\frac{1}{2}, \frac{1}{2})$, and the strategy is δ^0, the truncated δ^1. Notice that before the process terminates, δ^0 is either δ^1 or δ^2 depending on whether r is even or odd.

Since $A_1 = I$, $p_1 > p_2$, by the results for a stationary target model, an optimal strategy first searches box 1 to maximize $g^r(\mathbb{P}, A_1, \delta)$, where δ is any strategy. Since $(1 - \alpha)p_1 \leq p_2$, an optimal strategy next searches box 2. But $\alpha_1 = \alpha_2 = \alpha$ implies that after searching twice without finding it, the state becomes $\mathbb{P}$ again for the third stage. Therefore, repeating the above arguments shows that δ^1 is optimal for $g^r(\mathbb{P}, A_1, \delta)$ for any r.

Now $r \geq 1$ and $V = (\frac{1}{2}, \frac{1}{2}) = (v_1, v_2)$ satisfies $1 - \alpha \leq v_1/v_2 \leq 1/1 - \alpha$. Since $v_1 = v_2 = \frac{1}{2}$, by induction hypothesis, $g^{m-r}(V,T;\delta)$ is maximized by either δ^1 and δ^2. Hence δ^0 maximizes $g^{m-r}(V,T,\delta)$ for any r. It follows that δ^1 maximizes $g^m(\mathbb{P}, T, \delta)$ and the theorem is proven.

Q.E.D.

<u>Theorem 3.11</u>:

If $1 \leq \frac{a}{b} \leq \frac{1}{1 - \alpha}$, then for any state $\mathbb{P}$, an optimal strategy first searches a box with larger p_i.

<u>Proof</u>:

Theorem 3.9 says that if $\frac{p_1}{p_2} \geq \frac{a}{b}$ or if $\frac{p_1}{p_2} \leq \frac{b}{a}$, then an optimal strategy first searches a box with larger p_i. Theorem 3.10 says that

if $1 - \alpha \le \frac{p_1}{p_2} \le \frac{1}{1 - \alpha}$ then an optimal strategy first searches a box with larger p_i .

Now if $\frac{a}{b} \le \frac{1}{1 - \alpha}$, then $\frac{p_1}{p_2}$ satisfies the condition of either Theorem 3.9 or Theorem 10. Hence an optimal strategy first searches a box with larger p_i .

Remark:

Unfortunately, when $\frac{a}{b} > \frac{1}{1 - \alpha}$ and $\frac{1}{1 - \alpha} \le \frac{p_1}{p_2} \le \frac{a}{b}$, the optimal strategy is not characterized by the preceding theorems.

CHAPTER 4

OPTIMAL SEARCH WITH RANDOM OVERLOOK PROBABILITIES

4.1 Introduction

Consider an optimal search problem with n boxes. Let p_i be the given prior probability that the target is in box i, $p_i \geq 0$ $\sum_{i=1}^{n} p_i = 1$. The target is stationary. The overlook probabilities, however, are allowed to be random variables. Thus, searching box i at time t finds the target with probability $\alpha_i^t > 0$ if the target is in box i. For fixed i, $\alpha_i^1, \alpha_i^2, \ldots$ are independent identically distributed random variables. The α_i^t's are told at time t either before the search or after the search as will be treated separately in the sections that follow.

4.2 Random Overlook Probabilities Told After the Search

The main difference between this case and the model with deterministic random overlook probabilities is as follows. The posterior probabilities after a search is made without finding it are random variables. It follows that a strategy is usually not a fixed sequence of searching. In fact, the decision for time $t + 1$ is not made until α_i^t's are told which occurs after a search is completed at time t. A strategy δ, therefore, is any rule for determining which box to search at each time t. The rule depends on the posterior probability distribution at that time. Since for fixed i, α_i^t has the same distribution for all t, $E\alpha_i^t$ shall be written as $E\alpha_i$.

Theorem 4.1:

Let N be any integer, $\mathbb{P} = (p_1, p_2, \ldots, p_n)$ be the prior probabilities. To maximize the probability of finding the target in N

searches, an optimal strategy first searches the box with $\max_i p_i E\alpha_i$.

Proof:

Let N be the allowed number of searches.

Using induction, $N = 1$ is trivial. It will be verified that if the theorem holds for $N = m - 1$, then it holds for $N = m$.

In a m-stage problem, when the initial state is $\mathbb{P} = (p_1, \ldots, p_n)$, let box j be such that $p_j E\alpha_j = \max_i p_i E\alpha_i$. Suppose an optimal strategy first searches box k , $p_k E\alpha_k \neq \max p_i E\alpha_i$. After the search, if the target is not, found, then the posterior probability distribution $\mathbb{P}' = (p_1', p_2', \ldots, p_n')$ as a function of the told α_i's , is

$$p_k' = \frac{(1 - \alpha_k)p_k}{1 - \alpha_k p_k} \qquad p_i' = \frac{p_i}{1 - \alpha_k p_k} \qquad i \neq k .$$

$\mathbb{P}'$ is a random vector since α_k is a random variable. Moreover, $\mathbb{P}'$ can be considered as the initial state for the remaining $m - 1$ stage problem. Now

$$p_j E\alpha_j \geq p_i E\alpha_i \qquad i \neq j$$

$$\frac{p_k'}{p_k} \leq \frac{p_i'}{p_i} = \frac{1}{1 - \alpha_k p_k} \qquad \forall\ i \neq k .$$

Hence $p_j' E\alpha_j \geq p_i' E\alpha_i$ $\forall$ i . This is true regardless of the value of the α_i's .

By induction hypothesis, it will be optimal then to search box j first for the remaining $m - 1$ stage problem. Thus an optimal strategy

for the original m stage problem is to search first box k, then box j then continue optimally by following δ^*, say. Let the whole strategy be denoted by $S_k S_j \delta^*$.

For any strategy δ and any prior probability vector $\mathbb{P}$, let $f^m(\mathbb{P},\delta)$ = the probability of finding the ball in m searches when $\mathbb{P}$ is the vector of prior probabilities and strategy δ is employed.

If one can show

$$f^m\left(\mathbb{P}, S_j S_k \delta^*\right) - f^m\left(\mathbb{P}, S_k S_j \delta^*\right) \geq 0$$

then there exists an optimal strategy which first searches box j where $p_j E\alpha_j = \max_i \alpha_i p_i$. Let the α's for the second stage be α_i'. Let $T_j T_k p$ be the posterior probabilities after searching first box j then box k without finding the target. Then

$$f^m\left(\mathbb{P}, S_j S_k \delta^*\right) - f^m\left(\mathbb{P}, S_k S_j \delta^*\right) =$$

$$p_j E\alpha_j + p_k E\alpha_k' + E(1 - \alpha_j p_j)(1 - \alpha_k' p_k) f^{m-2}\left(T_j T_k \mathbb{P}, \delta^*\right) -$$

$$p_k E\alpha_k - p_j E\alpha_j' - E(1 - \alpha_k p_k)(1 - \alpha_j' p_j) f^{m-2}\left(T_k T_j \mathbb{P}, \delta^*\right)$$

where $T_j T_k \mathbb{P} = \dfrac{1}{(1 - \alpha_j p_j)(1 - \alpha_k' p_k)} (p_1, \ldots, (1 - \alpha_j)p_j, \ldots, (1 - \alpha_k')p_k,$

$$\ldots, p_n) .$$

By assumption, for any i, α_i and α_i' have the same distribution. Hence $f^m\left(\mathbb{P}, S_j S_k \delta^*\right) - f^m\left(\mathbb{P}, S_k S_j \delta^*\right) = 0$.

Q.E.D.

Assume searching box i costs c_i , $0 < c_i < \infty$. The problem of minimizing the expected searching cost will now be treated. For any strategy δ and any prior probability vector $\mathbb{P}$, let $g^m(\mathbb{P},\delta) = m$ stage expected searching cost when $\mathbb{P}$ is the vector or prior probabilities and strategy δ is employed.

Similarly, let $g(\mathbb{P},\delta)$ = total expected searching cost when $\mathbb{P}$ is the vector of prior probabilities and strategy δ is employed.

Define

$$g^m(\mathbb{P}) = \inf_\delta g^m(\mathbb{P},\delta)$$

$$g(\mathbb{P}) = \inf_\delta g(\mathbb{P},\delta) .$$

<u>Lemma 4.2</u>:

Let $\min_i c_i = c > 0$, $E\alpha_i > 0$, and $\max_i c_i = k < \infty$. Then

$$g(\mathbb{P}) \leq c \cdot k \Bigg/ \left(\sum_i \frac{c_i}{E\alpha_i} \right) = M(\text{say}) .$$

<u>Proof</u>:

Let δ^1 be the strategy of always searching the box with $\max_i \dfrac{p_i E\alpha_i}{c_i}$ at any time t . For any strategy δ , let $N^*(\delta)$ be the random time at which the target is found by using strategy δ . $N^*(\delta) = \infty$, if the target is never found. Then

$$g(\mathbb{P}) \leq g(\mathbb{P},\delta^1) \leq k \cdot EN^*(\delta^1)$$

$$EN^*(\delta^1) = \sum_{m=0}^{\infty} P(N^*(\delta^1) > m)$$

$$P(N^*(\delta^1) > m) = P_r\begin{pmatrix}\text{not finding the target} \\ \text{in } m \text{ searches by using} \\ \text{strategy } \delta^1 .\end{pmatrix}$$

The minimal value of $\max_i \frac{p_i E\alpha_i}{c_i}$ is achieved by the vector having

$$\frac{p_1 E\alpha_1}{c_1} = \frac{p_2 E\alpha_2}{c_2} = \dots = \frac{p_m E\alpha_m}{c_m} .$$

Now each time δ^1 searches a box with $\max_i \frac{p_i E\alpha_i}{c_i}$. Thus, each time δ^1 searches a box (say box j), the probability $p_j E\alpha_j$ that the target will be found is such that

$$p_j E\alpha_j \geq c_j \Big/ \left(\sum_i \frac{c_i}{E\alpha_i}\right) \geq c \Big/ \left(\sum_i \frac{c_i}{E\alpha_i}\right) .$$

Hence

$$P(N^*(\delta^1) > m) \leq \left[1 - c \Big/ \left(\sum_i \frac{c_i}{E\alpha_i}\right)\right]^m$$

$$EN^*(\delta^1) = \sum_{m=0}^{\infty} P(N^*(\delta^1) > m) \leq c \Big/ \left(\sum_i \frac{c_i}{E\alpha_i}\right)$$

$$g(\mathbb{P}) \leq k \cdot EN^*(\delta^1) \leq c \cdot k \Big/ \left(\sum_i \frac{c_i}{E\alpha_i}\right) .$$

Q.E.D.

<u>Theorem 4.2</u>:

Let $g^m(\mathbb{P})$, $g(\mathbb{P})$ be defined as before, then

$$\lim_{m \to \infty} g^m(\mathbb{P}) = g(\mathbb{P}) .$$

Proof:

Let δ^m be a strategy which minimizes the m stage expected searching cost. Let $g^m(\mathbb{P},\delta)$ be the m stage expected cost and let $g(\mathbb{P},\delta)$ be the total expected searching cost defined as before. $g^m(\mathbb{P},\delta)$ is a monotone increasing function of m. The same is true for $g^m(\mathbb{P})$. Since $g^m(\mathbb{P}) \leq g(\mathbb{P}) \leq M$ (a constant), $g^m(\mathbb{P})$ converges in m. Let $P^m(\delta)$ be the probability of finding the target in m searches by following strategy δ. Then

$$g(\mathbb{P},\delta^m) = g^m(\mathbb{P},\delta^m) + [1 - P^m(\delta^m)]g(T^m\mathbb{P})$$

where $T^m\mathbb{P}$ is the posterior probability after using δ^m for m stages without finding the target. Now $M \geq g(T^m\mathbb{P})$ and $P^m(\delta^m) \to 1$ (otherwise $g^m(\mathbb{P}) \to \infty$). Hence, $g(\mathbb{P},\delta^m) \to g^m(\mathbb{P},\delta^m) = g^m(\mathbb{P})$. Suppose $g^m(\mathbb{P}) \to K < g(\mathbb{P})$. Then for N large enough, $g(\mathbb{P},\delta^N) < g(\mathbb{P})$ which is a contradiction. Hence, $\lim_{m\to\infty} g^m(\mathbb{P}) = g(\mathbb{P})$.

Q.E.D.

Theorem 4.3:

Let $\mathbb{P} = (p_1, \ldots, p_n)$ be the state at a certain time. To minimize the expected searching cost, an optimal strategy first searches a box with $\max \frac{p_i E\alpha_i}{c_i}$.

Proof:

The proof will be carried out by considering an m stage searching process and then let m go to infinity. Let initial state $\mathbb{P}$ be as

given. Let m be any positive integer. Consider an m stage searching process. Let box j be a box with $\max \frac{p_i E\alpha_i}{c_i}$.

Define the following strategies:

δ^1 = the strategy which minimizes the m stage expected searching cost given that it searches box j at the mth stage.

δ^2 = an optimal strategy which minimizes the m stage expected searching cost.

δ^3 = the strategy which first searches box j and then continues optimally.

For any strategy δ , let $g^m(\mathbb{P},\delta)$ be defined as the m stage expected cost and let $g^m(\mathbb{P}) = \inf_\delta g^m(\mathbb{P},\delta)$ as before. Also let $g(\mathbb{P})$ be the minimum expected cost before finding the target $(m = \infty)$.

By Theorem 4.2, for any state $\mathbb{P}$,

$$g^m(\mathbb{P}) \to g(\mathbb{P}) .$$

Recall that $T_i\mathbb{P}$ is the posterior probability for the next stage after searching box i without finding the target. Then by definition,

$$g^m(\mathbb{P},\delta^3) = c_j + E(1 - \alpha_j p_j) g^{m-1}(T_j\mathbb{P})$$

$$g^m(\mathbb{P},\delta^3) \to c_j + E(1 - \alpha_j p_j) g(T_j\mathbb{P}) \quad \text{as} \quad m \to \infty .$$

Now by dynamic programming,

$$g(\mathbb{P}) = \min_i c_i + E(1 - \alpha_i p_i) g(T_i\mathbb{P}) .$$

Hence if one can show $g^m(\mathbb{P}, \delta^3) \to g(\mathbb{P})$ then an optimal strategy first searches box j for an infinite stage process.

In order to show $g^m(\mathbb{P}, \delta^3) \to g(\mathbb{P})$, it suffices to prove the following two parts:

(a) $g^m(\mathbb{P}, \delta^3) \leq g^m(\mathbb{P}, \delta^1)$.

(b) As $m \to \infty$, $g^m(\mathbb{P}, \delta^1) \to g^m(\mathbb{P}, \delta^2) \to g(\mathbb{P})$.

To prove (a), induction will be used. When $m = 1$, $g^m(\mathbb{P}, \delta^3) = g^m(\mathbb{P}, \delta^1) = c_j$, (a) is trivially true. It will be verified that if (a) holds for $m = r - 1$ for any $\mathbb{P}$, then it holds for $m = r$ as well.

Loosely speaking, $g^m(\mathbb{P}, \delta^3) \leq g^m(\mathbb{P}, \delta^1)$ means that if box j has $\max \frac{p_i E\alpha_i}{c_i}$ then searching box j first is no worse than searching box j last, when optimal decision is made at the other stages.

Suppose when $m = r$, δ^1 searches box k first. If $k = j$, the case is trivial. So assume $k \neq j$. After the first search, if the target is not found, let the posterior probability be $\mathbb{P}' = (p_1', p_2', \ldots, p_n')$. Let α_i^t be defined as before. To simplify notation, let $\alpha_i^1 = \alpha_i$, $\alpha_i^2 = \alpha_i'$, $i = 1, \ldots, n$. Then, for any i

$$\frac{p_i'}{p_i} = \frac{1}{1 - \alpha_k p_k} \quad \text{if} \quad i \neq k$$

$$\frac{p_i'}{p_i} = \frac{1 - \alpha_k}{1 - \alpha_k p_k} \quad \text{if} \quad i = k .$$

Hence $\frac{p_j' E\alpha_j}{c_j} = \max\limits_i \frac{p_i' E\alpha_i}{c_i}$ where $\mathbb{P}'$ is the state for the remaining

$r - 1$ stages. By induction hypothesis, searching box j next is no worse than searching box j last.

Let $S_k S_j \delta^*$ be the strategy that searches first box k, then box j then follows an optimal strategy δ^*. Then $S_k S_j \delta^*$ is no worse than δ^1 for the r stage process, i.e.,

$$g^r\left(\mathbb{P}, S_k S_j \delta^*\right) \le g^r(\mathbb{P}, \delta^1) .$$

Let $S_j S_k \delta^*$ be similarly defined. If one can show

$$g^r\left(\mathbb{P}, S_j S_k \delta^*\right) - g^r\left(\mathbb{P}, S_k S_j \delta^*\right) \le 0$$

then $g^r(\mathbb{P}, \delta^3) \le g^r\left(\mathbb{P}, S_j S_k \delta^*\right) \le g^r\left(\mathbb{P}, S_k S_j \delta^*\right) \le g^r(\mathbb{P}, \delta^1)$ and (a) will be proven. Now

$$g^r\left(\mathbb{P}, S_j S_k \delta^*\right) - g^r\left(\mathbb{P}, S_k S_j \delta^*\right) =$$

$$c_j + (1 - p_j E\alpha_j) c_k + E(1 - \alpha_j p_j)(1 - \alpha_k' p_k') g^{r-2}(T_j T_k \mathbb{P}) -$$

$$c_k - (1 - p_k E\alpha_k) c_j - E(1 - \alpha_k p_k)(1 - \alpha_j' p_j') g^{r-2}(T_k T_j \mathbb{P})$$

where $T_j T_k \mathbb{P}$ is the posterior probability after searching first box j then box k without finding the target. Following the arguments as used in the proof of Theorem 4.1 yields

$$g^m\left(\mathbb{P}, S_j S_k \delta^*\right) - g^m\left(\mathbb{P}, S_k S_j \delta^*\right) =$$

$$-(p_j E\alpha_j) c_k + (p_k E\alpha_k) c_j =$$

$$c_k c_j \left(\frac{p_k E\alpha_k}{c_k} - \frac{p_j E\alpha_j}{c_j}\right) \le 0 .$$

Hence (a) is proven.

The (b) part is repeated here to be proven, $g^m(\mathbb{P},\delta^1) \to g^m(\mathbb{P},\delta^2) \to g(\mathbb{P})$ as $m \to \infty$. For any strategy δ, any integer N, let $f^N(\mathbb{P},\delta)$ be the probability of finding the target in N searches when $\mathbb{P}$ is the initial state and strategy δ is employed. Then by definition of δ^1, δ^2,

$$g^m(\mathbb{P},\delta^1) = \inf_{\delta} \left\{g^{m-1}(\mathbb{P},\delta) + [1 - f^{m-1}(\mathbb{P},\delta)]\cdot c_j\right\} \leq$$

$$g^{m-1}(\mathbb{P},\delta^2) + [1 - f^{m-1}(\mathbb{P},\delta^2)]\cdot \max c_i$$

$$g^m(\mathbb{P},\delta^2) = g^{m-1}(\mathbb{P},\delta^2) + [1 - f^{m-1}(\mathbb{P},\delta^2)]\cdot \min c_i .$$

Hence

$$0 \leq g^m(\mathbb{P},\delta^1) - g^m(\mathbb{P},\delta^2) \leq [1 - f^{m-1}(\mathbb{P},\delta^2)]\cdot[\max c_i - \min c_i] .$$

By Lemma 4.2, $g(\mathbb{P})$ is bounded. Hence $g^m(\mathbb{P}) = g^m(\mathbb{P},\delta^2)$ is bounded. If, as $m \to \infty$, $f^{m-1}(\mathbb{P},\delta^2) \not\to 1$, then there is a finite probability that the target will never be found. Since $\min_i c_i > 0$, this would imply $g^m(\mathbb{P},\delta^2)$ becomes unbounded, which is a contradiction. Therefore

$$g^m(\mathbb{P},\delta^1) \to g^m(\mathbb{P}) \quad \text{and}$$

$$g^m(\mathbb{P}) \to g(\mathbb{P}) \quad \text{by Theorem 4.2.}$$

Q.E.D.

4.3 Random Overlook Probabilities Told before the Search

In this section, the case where the α's are told before the search will be analyzed. Thus let α_i^t be the probability of finding

the target when a search is made in box i at time t, given that the target is in box i. Again, for any fixed i, $\alpha_i^1, \alpha_i^2, \ldots$ are independent identically distributed random variables. The fact that α_i^t's are told before time t makes it necessary to include the α_i^t's as part of the state at time t. That is, the state at time t now consists of the posterior probability of the target being in box i at time t, (call it p_i^t) as well as the α_i^t's.

To simplify the discussion, two types of α_i^t's are chosen. One is the case where for any fixed t, $\alpha_1^t, \alpha_2^t, \ldots, \alpha_n^t$ are assumed to be independent random variables. The other is the case where $\alpha_i^t \equiv \alpha^t$, i.e., at any time t, α_i^t's are identical for all the boxes.

Consider first the case where for fixed t, $\alpha_1^t, \alpha_2^t, \ldots, \alpha_n^t$ are independent random variables. In the problem of maximizing the probability of finding the target in a given number of searches, one might conjecture that an optimal strategy searches the box with $\max_i \alpha_i p_i$ each time. The following counterexample shows that this is not always true.

Suppose in a two box optimal search problem, the objective is to maximize the probability of finding the target in two stages. Let the prior probabilities at the first stage be p_1, p_2. For any t, let α_1^t, α_2^t have the following probability distribution.

$$\alpha_1 = \begin{cases} 1 & \text{with probability } a \\ 0 & \text{with probability } 1-a \end{cases}$$

$$\alpha_2 = \begin{cases} 1 & \text{with probability } b \\ 0 & \text{with probability } 1-b\,. \end{cases}$$

Assume that at the beginning of the first stage, α_1 and α_2 are told to be $\alpha_1 = \alpha_2 = 1$, while $p_1 > p_2 > 0$. Then according to the conjecture, an optimal strategy searches box 1 first. Moreover, since $\alpha_1 = 1$ initially, if the ball is not found at the first search, then the ball is not in box 1 and one always searches box 2 at the next stage. Let the α's for the second stage be α_1' and α_2' and consider the following two strategies. One is to search first box 1 then box 2, call this strategy S_1S_2 . The other is to search first box 2 then box 1, call it S_2S_1 . The probability of finding the target by using the first strategy is $p_1 + p_2E\alpha_2'$. The same probability by using the second strategy is $p_2 + p_1E\alpha_1'$. Clearly, if $p_2(1 - E\alpha_2') > p_1(1 - E\alpha_1')$, then S_1S_2 is not optimal and the conjecture is wrong. It is easy to see that for some suitably chosen a and b in the probability distribution of α , namely, for $p_2(1 - b) > p_1(1 - a)$, the counterexample is established.

The following definitions will be used in the theorems that come later.

Let $(\mathbb{P},\alpha)$ be the state at a certain stage, where $\mathbb{P} = (p_1,p_2, \ldots, p_n)$ is the posterior probability vector at that stage and $\alpha = (\alpha_1,\alpha_2, \ldots, \alpha_n)$ is the α_i^t's at that stage. The superscript t in α_i^t is suppressed since the time is understood to be that specified stage.

A strategy δ is defined to be any rule for determining which box to search at a given state. Since the prior probabilities are given, a strategy δ yields a searching sequence which depends on the α_i^t's which are told each time before the search.

For any strategy δ, any integer m and any initial state $(\mathbb{P},\alpha)$, let $f^m(\mathbb{P},\alpha;\delta)$ = the probability of finding the target in m searches

when $(\mathbb{P},\alpha)$ is the state at the first stage and strategy δ is employed thereafter.

Let $f^m(\mathbb{P},\alpha) = \inf_\delta f^m(\mathbb{P},\alpha;\delta)$. For any i , let $f_i^m(\delta)$ = the conditional probability of finding the target in m searches given that the target is in box i and strategy δ is employed.

Then $Ef^m(\mathbb{P},\alpha;\delta) = \sum_{i=1}^{n} p_i f_i^m(\delta)$.

Theorem 4.4:

Let box i be any box, $(\mathbb{P},\alpha)$ be any state. Let $\alpha' = (\alpha_1',\alpha_2', \ldots, \alpha_n')$ be another α vector such that

$$\alpha_i' \geq \alpha_i \ , \ \alpha_k' \leq \alpha_k \qquad \forall\ k \neq i \ .$$

Suppose an optimal strategy first searches box i at state $(\mathbb{P},\alpha)$. Then an optimal strategy first searches box i at state $(\mathbb{P},\alpha')$ as well.

Proof:

For a given initial state $(\mathbb{P},\alpha)$, and for any j , let $T_j\mathbb{P}$ be the posterior probability vector for the next stage given that the present search of box j has not uncovered the target. Thus

$$T_j\mathbb{P} = [(T_j\mathbb{P})_1, \ldots, (T_j\mathbb{P})_n] \text{ , where}$$

$$(T_j\mathbb{P})_r = \begin{cases} p_r(1 - \alpha_j p_j)^{-1} & (r \neq j) \\ (1 - \alpha_j)p_j(1 - \alpha_j p_j)^{-1} & (r = j) \ . \end{cases}$$

Let box i be the box specified above.

For the initial state $(\mathbb{P},\alpha)$, let $S_i\delta^*$ be the strategy of first searching box i then following an optimal strategy δ^* . Let box k

be any box. For the initial state $(\mathbb{P},\alpha')$, let $S_k\delta'$ be the strategy of first searching box k , $k \neq i$ then following an optimal strategy δ' . Let α^2 be the α vector for the next stage. Then

$$f^m(\mathbb{P},\alpha) = f^m\left(\mathbb{P},\alpha;S_i\delta^*\right) = \alpha_i p_i + (1-\alpha_i p_i)Ef^{m-1}\left(T_i\mathbb{P},\alpha^2\right) =$$

$$\alpha_i p_i + (1-\alpha_i p_i)\sum_{j=1}^{n}(T_i\mathbb{P})_j f_j^{m-1}(\delta^*) =$$

$$\alpha_i p_i + (1-\alpha_i)p_i f_i^{m-1}(\delta^*) + \sum_{j\neq i} p_j f_j^{m-1}(\delta^*) =$$

$$\alpha_i p_i\left[1 - f_i^{m-1}(\delta^*)\right] + \sum_{j=1}^{n} p_j f_j^{m-1}(\delta^*) \ .$$

Since $\alpha_i' \geq \alpha_i$, $\alpha_k' \leq \alpha_k$

$$f^m\left(\mathbb{P},\alpha';S_i\delta^*\right) = \alpha_i' p_i\left[1 - f_i^{m-1}(\delta^*)\right] + \sum_{k=1}^{n} p_k f_k^{m-1}(\delta^*) \geq$$

$$f^m\left(\mathbb{P},\alpha,S_i\delta^*\right) = f^m(\mathbb{P},\alpha) \geq f^m(\mathbb{P},\alpha,S_k\delta') =$$

$$\alpha_k p_k\left[1 - f_k^{m-1}(\delta')\right] + \sum_{j=1}^{n} p_j f_j^{m-1}(\delta') \geq$$

$$\alpha_k' p_k\left[1 - f_k^{m-1}(\delta')\right] + \sum_{j=1}^{n} p_j f_j^{m-1}(\delta') =$$

$$f^m(\mathbb{P},\alpha';S_k\delta') = \alpha_k' p_k + (1-\alpha_k' p_k)Ef^{m-1}\left(T_k\mathbb{P},\alpha^2\right) \ .$$

Hence an optimal strategy first searches box i at state $(\mathbb{P},\alpha')$.

Q.E.D.

Assume searching any box i costs $c_i > 0$. Now consider the problem of minimizing the expected searching cost. For any strategy δ ,

any initial state $(\mathbb{P},\alpha)$, let $g(\mathbb{P},\alpha;\delta)$ = the expected searching cost when $(\mathbb{P},\alpha)$ is the state at the first stage and strategy δ is employed thereafter.

Let $g(\mathbb{P},\alpha) = \inf_{\delta} g(\mathbb{P},\alpha;\delta)$. For any box i, let $g_i(\delta)$ = the conditional expected searching cost given that the target is in box i and strategy δ is employed.

Then $Eg(\mathbb{P},\alpha;\delta) = \sum_{i=1}^{n} p_i g_i(\delta)$.

<u>Theorem 4.5</u>:

Let box i be any box, $(\mathbb{P},\alpha)$ be any state. Let $\alpha' = (\alpha'_1,\alpha'_2, \ldots, \alpha'_n)$ be another α vector such that

$$\alpha'_i \geq \alpha_i \ , \ \alpha'_k \leq \alpha_k \qquad \forall\ k \neq i \ .$$

Suppose an optimal strategy first searches box i at state $(\mathbb{P},\alpha)$. Then an optimal strategy first searches box i at state $(\mathbb{P},\alpha')$ as well.

<u>Proof:</u>

Let $T_j\mathbb{P}$ be defined as before. For the initial state $(\mathbb{P},\alpha)$, let $S_i\delta^*$ be the strategy of first searching box i then following an optimal strategy δ^*. For the initial state $(\mathbb{P},\alpha')$, let $S_k\delta'$ be the strategy of first searching box k, $k \neq i$, then following an optimal strategy δ'. Let α^2 be the α vector for the next stage. Then

$$g(\mathbb{P},\alpha) = g\left(\mathbb{P},\alpha,S_i\delta^*\right) = c_i + (1-\alpha_i p_i)Eg\left(T_i\mathbb{P},\alpha^2\right) =$$

$$c_i + (1-\alpha_i p_i)\sum_{j=1}^{n}(T_i\mathbb{P})_j g_j(\delta^*) =$$

$$c_i + (1-\alpha_i)p_i g_i(\delta^*) + \sum_{j\neq i} p_j g_j(\delta^*) =$$

$$c_i - \alpha_i p_i g_i(\delta^*) + \sum_{j=1}^{n} p_j g_j(\delta^*) \ .$$

Since $\alpha_i' \geq \alpha_i$, $\alpha_k' \leq \alpha_k$

$$g\left(\mathbb{P},\alpha';S_i\delta^*\right) = c_i - \alpha_i' p_i g_i(\delta^*) + \sum_{j=1}^{n} p_j g_j(\delta^*) \leq$$

$$g\left(\mathbb{P},\alpha;S_i\delta^*\right) = g(\mathbb{P},\alpha) \leq g(\mathbb{P},\alpha;S_k\delta') =$$

$$c_k - \alpha_k p_k g_k(\delta') + \sum_{j=1}^{n} p_j g_j(\delta') \leq g(\mathbb{P},\alpha';S_k\delta') \ .$$

Hence an optimal strategy first searches box i at state $(\mathbb{P},\alpha')$ as well.

Q.E.D.

Consider the case where for fixed time t , $\alpha_i^t \equiv \alpha^t$, i.e., all the boxes have the same overlook probabilities. In the problem of maximizing the probability of finding the target in a given number of searches, one might conjecture again that an optimal strategy first searches the box with $\max_i p_i$. The conjecture is not always true. Another conjecture is that if $\mathbb{P} = (p_1, \ldots, p_n)$, α is given and it is optimal to first search box i at state $(\mathbb{P},\alpha)$ then it is also optimal to first search box i at state $(\mathbb{P}',\alpha)$ where $\mathbb{P}' = (p_1', \ldots, p_n')$, $p_i' \geq p_i$, $p_k' \leq p_k$, $k \neq i$. Note that the first conjecture implies the second one. A counterexample is given below to show that even the second conjecture is not always true.

Let there be two boxes. Consider a two stage optimal search problem. The objective is to maximize the probability of finding the target in two searches. Notice that $\alpha_1^t = \alpha_2^t = \alpha^t$ in this case. Suppose $\alpha^1 = \alpha$, $\alpha^2 = \alpha'$, the prior probability vector is $\mathbb{P} = (p_1, p_2)$. For the last stage (the second stage), obviously the box with the larger probability of containing the target ought to be searched. If the second conjecture were true, then there would be a number $\lambda \geq 0$ such that an optimal strategy first searches box 1 when $\frac{p_1}{p_2} \geq \lambda$, and first searches box 2 when $\frac{p_1}{p_2} \leq \lambda$. This will be disproved.

Assume $\alpha < E\alpha'$ and assume first $(1 - \alpha)p_1 \geq p_2$. The probability of finding the target by searching first box 1 then the box with larger posterior probability of containing the target is

$$\alpha p_1 + (1 - \alpha p_1)\cdot(E\alpha')\cdot \max_{1,2} \left\{\frac{(1 - \alpha)p_1}{1 - \alpha p_1} , \frac{p_2}{1 - \alpha p_2}\right\}$$

$$(*) \qquad = \alpha p_1 + (E\alpha')\cdot \max_{1,2} \{(1 - \alpha)p_1, p_2\} =$$

$$\alpha p_1 + (E\alpha')\cdot(1 - \alpha)p_1$$

since $(1 - \alpha)p_1 \geq p_2$, by assumption. The probability defined the same way as above except that box 2 is searched first is

$$\alpha p_2 + (E\alpha')\cdot \max_{1,2} \{p_1, (1 - \alpha)p_2\} =$$

$$(**) \qquad \alpha p_2 + (E\alpha')\cdot p_1$$

since $p_1 \geq (1 - \alpha)p_1 \geq p_2 \geq (1 - \alpha)p_2$. Subtracting (**) from (*) yields

$$\alpha[p_1(1 - E\alpha') - p_2] .$$

So if $p_1(1 - \alpha) \geq p_2$, an optimal strategy first searches box 1 or box 2 depending on $p_1(1 - E\alpha') - p_2$ is positive or negative.

By symmetry, if $p_2(1 - \alpha) \geq p_1$, an optimal strategy first searches box 2 or box 1 depending on $p_2(1 - E\alpha') - p_1$ is positive or negative. It follows that under the assumption that $\alpha < E\alpha'$, the optimal strategy searches box 1 when $\frac{p_1}{p_2} \geq \frac{1}{1 - E\alpha'}$ or when $1 - \alpha \geq \frac{p_1}{p_2} \geq 1 - E\alpha'$ and searches box 2 when $\frac{1}{1 - \alpha} \leq \frac{p_1}{p_2} \leq \frac{1}{1 - E\alpha'}$ or when $\frac{p_1}{p_2} \leq 1 - E\alpha'$.

Hence the second conjecture was wrong.

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